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AN ACCURATE METHOD FOR FREE VIBRATION ANALYSIS OF STRUCTURES WITH APPLICATION TO PLATES

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In this work, the continuous element method which has been used as an alternative to the finite element method of vibration analysis of frames is applied to more general structures like 3-D continuum and rectangular plates. The method is based on the concept of the so-called impedance matrix giving in the frequency domain, the linear relation between the generalized displacements of the boundaries and the generalized forces exerted on these boundaries. For a 3-D continuum, the concept of impedance matrix is introduced assuming a particular kind of boundary conditions. For rectangular plates, this new development leads to the solution of vibration problems for boundary conditions other than the simply supported ones.

1. INTRODUCTION

The concept of using an impedance matrix or dynamic stiffness matrix is well known in the dynamic analysis of structures. The traditional finite element method [1] of modal analysis leads to errors due to the spatial discretization of the structure. The accuracy of results depends on the number of finite elements used in the mesh, but the increase in element number becomes expensive in computer time. It is well known that when using the finite element method, the accuracy in mode shapes is not as good as in natural frequencies. An alternative to these modal methods is the so-called distributed or continuous element method, based on the exact solution of the partial differential equations describing the system. The distributed element method leads to the use of the impedance matrix of the non-discretized structure, in contrast with the finite element method. This continuous method allows us to obtain the eigensolutions in any frequency bandwidth, whereas the finite element method usually gives only a good accuracy for low frequencies.

In the past, the continuous element method was used only for one-dimensional bodies, like beams, extensible strings and rods [2, 3]. An extension for a 2-D structure idealized by a rectangular membrane under uniform tension can be found in reference [4].

In these cases, the impedance matrix, $\mathbf{Z}(\omega^2)$, which gives in frequency domain the correspondence between forces and torques exerted on the boundaries and displacements of these boundaries, can be obtained in analytical form.

A great amount of relevant publications since 1969 about dynamic stiffness analysis for vibrations of beam–column structures was undertaken initially by Wittrick [5] and then by Williams [6]. A review of the works done which include several algorithms for finding the

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modal values of a transcendental problem and the corresponding modal vectors with special attention for spinning and repetitive structures can be found in reference [7]. The papers also include several extensions to prismatic isotropic and anisotropic plate assemblies [8]. However, the analysis is only applicable to structures which are simply supported at their two ends or for an infinitely long plate assembly. Spectral methods for the vibration analysis of plates can be found in reference [9].

An extension of the continuous element method for a 3-D structure can be found in reference [10], assuming some particular kind of boundary conditions. The corresponding impedance matrix, $\mathbf{Z}(\omega^2)$, is a symmetric matrix depending on the frequency for which the well-known Leung's theorem [11] can be applied as follows:

$$\mathbf{M}(\omega^2) = -\frac{\mathrm{d}}{\mathrm{d}\omega^2}\mathbf{Z}(\omega^2). \tag{1}$$

In equation (1), $\mathbf{M}(\omega^2)$ is the mass matrix of the structure and ω is the circular frequency. Some extensions of the method introduced in reference [10] can be found in reference [12]. For a general 3-D continuum, no closed-form expression for the impedance matrix is available but it is possible to obtain a spectral expansion of the matrix in terms of some set of modal frequencies of the structure [10, 12].

Other attempts [13–15], to extend the distributed element method for plates and shells have been carried out. However, for these structures, the concept of using an impedance matrix is not well defined, leading in some cases [13, 15], to a non-symmetric matrix. In this work the method used for a 3-D structure in reference [10] is applied to rectangular plates in order to perform accurate modal analysis. This new development leads to the solution of vibration problem for boundary conditions other than the simply supported ones.

2. IMPEDANCE MATRIX FOR DISCRETIZED STRUCTURES AND ONE-DIMENSIONAL BODIES

For a discretized system, described by its stiffness matrix $\mathbf{K}$ and its mass matrix $\mathbf{M}$, the impedance matrix, $\mathbf{Z}(\omega^2)$, is the frequency-dependent matrix defined by

$$\mathbf{Z}(\omega^2) = \mathbf{K} - \omega^2\mathbf{M}, \tag{2}$$

which relates the applied force amplitudes $\mathbf{Q}$ to the displacement amplitudes $\mathbf{q}$. The reduced impedance matrix of the structure is obtained by splitting the degrees of freedom $\mathbf{q}$ into internal degrees of freedom $\mathbf{q}_1$ and boundary degrees of freedom $\mathbf{q}_2$. Assuming that the internal degrees of freedom are free, the linear system

$$\mathbf{Q} = \mathbf{Z}(\omega^2)\,q, \qquad \mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2)^t, \qquad \mathbf{Q} = (\mathbf{0}, \mathbf{Q}_2)^t, \qquad \mathbf{Z} = \begin{pmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{12}^t & \mathbf{Z}_{22} \end{pmatrix} \tag{3}$$

can be transformed to $\mathbf{Q}_2 = \mathbf{Z}_c(\omega^2)\,\mathbf{q}_2$, where the reduced impedance matrix is given by

$$\mathbf{Z}_c = \mathbf{Z}_{22} - \mathbf{Z}_{12}^t\mathbf{Z}_{11}^{-1}\mathbf{Z}_{12}. \tag{4}$$

This reduced impedance matrix gives, in the frequency domain, the amplitudes of the forces exerted on the boundaries in terms of the displacements of the boundaries. In this case, the reduced mass matrix, $\mathbf{M}_c$, of the structure is related to the reduced impedance matrix $\mathbf{Z}_c$ by

$$\mathbf{M}_c(\omega^2) = -(\mathrm{d}/\mathrm{d}\omega^2)\mathbf{Z}_c(\omega^2). \tag{5}$$

For a one-dimensional continuum idealized by extensional bars or beams in bending or torsional vibrations, it is possible to define, in the frequency domain, the impedance matrix, $\mathbf{Z}(\omega^2)$, giving the correspondence between the forces and the torques exerted on the boundaries in terms of the displacements of these boundaries. The impedance matrix is obtained in an analytical form for a homogeneous and constant section beam [16]. Several codes [3, 17], which use the concept of continuous element, have been developed to perform accurate modal analysis of frames. In this case, the impedance matrix, $\mathbf{Z}(\omega^2)$, is a symmetric matrix which is a transcendental function of ω^2, instead of a rational function as in the case of a discretized structure. However, a very interesting result, obtained by Leung [16], is

$$\mathbf{M}(\omega^2) = -(\mathrm{d}/\mathrm{d}\omega^2)\,\mathbf{Z}(\omega^2),$$

where $\mathbf{M}(\omega^2)$ is the so-called dynamic mass matrix which is the symmetric matrix occurring in the expression of the kinetic energy of the system.

For a system composed of extensional rods or beams, it is possible to perform an accurate modal analysis by solving the system

$$\mathbf{Z}(\omega^2)\,\mathbf{X} = 0, \tag{6}$$

where $\mathbf{X}$ are the eigenvectors and the corresponding frequencies are the roots of the transcendental equation

$$\det(\mathbf{Z}(\omega^2)) = 0. \tag{7}$$

The standard methods of modal analysis are not valid in this case. Special algorithms have been used to solve this problem, using the well-known Williams and Wittrick algorithm [5], giving in any bandwidth of frequencies the number of roots of equation (7).

3. THREE-DIMENSIONAL CONTINUUM

For a 3-D continuum, the concept of the impedance matrix has been introduced in [10] and extended in reference [12]. In the following part, the main steps of the method are recalled.

Considering a 3-D flexible body (Figure 1) free of external body forces. Assume that on one part (Γ_1), of the boundary of the domain (V) occupied by the body, homogeneous boundary conditions occur, and on the other part, (Γ_2), of the boundary, are assumed

$$\vec{\mathbf{u}} = \vec{\mathbf{u}}_\Gamma = \vec{\mathbf{N}}(\vec{\mathbf{x}})\tilde{\mathbf{q}}(t), \qquad \tilde{\mathbf{q}} = [\tilde{q}_1, \ldots, \tilde{q}_p], \qquad \vec{\mathbf{N}} = [\vec{\mathbf{N}}_1, \ldots, \vec{\mathbf{N}}_p]. \tag{8}$$

where the imposed displacement field $\vec{\mathbf{u}}_\Gamma$ is a known function of a finite number of displacement parameters $\tilde{\mathbf{q}}(t)$; $\vec{\mathbf{x}} = \overrightarrow{\mathbf{OM}}$ gives the position with respect to the origin O of any material point M in (V).

By supposing that the assumed displacements on the boundary (Γ_2) are harmonic functions of time, $\tilde{\mathbf{q}} = \mathbf{q}\mathrm{e}^{\mathrm{i}\omega t}$, in the frequency domain, the motion of the body is defined by

$$\mathrm{L}[\vec{\mathbf{U}}] + \rho\omega^2\vec{\mathbf{U}} = 0 \quad \mathrm{in}\,(V), \qquad \vec{\mathbf{U}} = \vec{\mathbf{N}}\mathbf{q} \quad \mathrm{on}\,(\Gamma_2), \qquad \vec{\mathbf{U}} = \vec{\mathbf{0}} \quad \text{or} \quad \vec{\mathbf{F}} = \boldsymbol{\sigma}\vec{\mathbf{n}} = \vec{\mathbf{0}} \quad \mathrm{on}\,(\Gamma_1). \tag{9}$$

In this formula, $\vec{\mathbf{u}} = \vec{\mathbf{U}}\mathrm{e}^{\mathrm{i}\omega t}$, $\tilde{\boldsymbol{\sigma}}$ is the stress tensor ($\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}\mathrm{e}^{\mathrm{i}\omega t}$), ρ is the mass density, $\vec{\mathbf{n}}$ is the unitarian external vector normal to (Γ_1) and L is the usual linear self-adjoint differential operator of the classical elasticity.

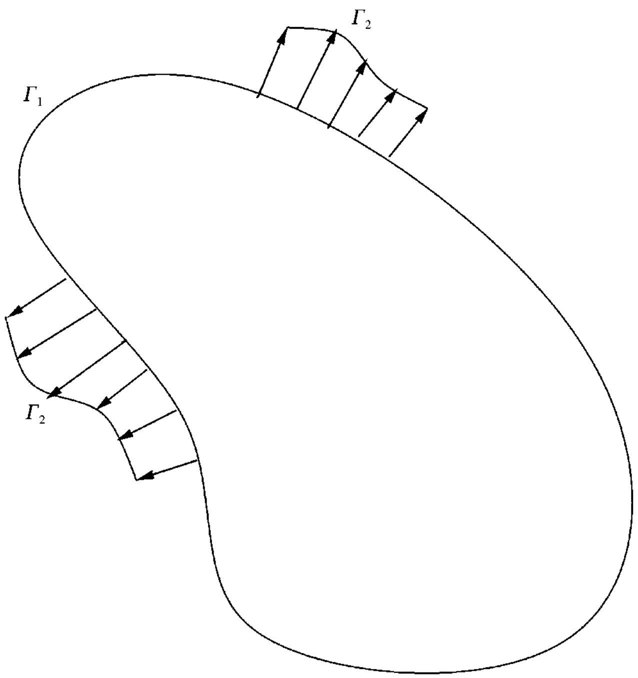


Figure 1. Three-dimensional continuum.

The solution of the linear problem (9) can be obtained in the form

$$\vec{\mathbf{U}} = \vec{\mathbf{P}}(\vec{\mathbf{x}}, \omega^2)\mathbf{q} \quad \text{in}\,(V), \qquad \vec{\mathbf{P}} = [\vec{\mathbf{P}}_1, \dots, \vec{\mathbf{P}}_p], \qquad \vec{\mathbf{P}} = \vec{\mathbf{N}} \quad \text{on}\,(\Gamma_2). \tag{10}$$

The corresponding surface forces $\vec{\mathbf{F}}_2$ applied on the boundary (Γ_2) is a linear function of $\mathbf{q}$

$$\vec{\mathbf{F}}_2 = \vec{\mathbf{\Phi}}(\vec{\mathbf{x}})\,\mathbf{q}, \qquad \vec{\mathbf{\Phi}} = [\vec{\mathbf{\Phi}}_1, \dots, \vec{\mathbf{\Phi}}_p]. \tag{11}$$

By introducing the generalized forces $\mathbf{Q}$ conjugated to $\mathbf{q}$, the work T done by the surface force $\vec{\mathbf{F}}_2$ can be computed from

$$T = \int_{\Gamma_2} \vec{\mathbf{F}}_2 \cdot \vec{\mathbf{U}}\,\mathrm{d}S = \int_{\Gamma_2} (\vec{\mathbf{F}}_2 \cdot \vec{\mathbf{N}})\,\mathrm{d}S\mathbf{q} = \mathbf{Q}^t\mathbf{q}, \qquad \mathbf{Q} = \int_{\Gamma_2} \vec{\mathbf{N}}^t \cdot \vec{\mathbf{F}}_2\,\mathrm{d}S = \mathbf{Z}\mathbf{q}, \tag{12}$$

where $\mathbf{Z} = \int_{\Gamma_2} \vec{\mathbf{N}}^t \cdot \vec{\mathbf{\Phi}}\,\mathrm{d}S$ is the $(p \times p)$ frequency-dependent impedance matrix, giving the linear relation between the generalized displacements $\mathbf{q}$ and the corresponding generalized forces $\mathbf{Q}$.

Using the Green formula and taking into account the properties of the differential operator $\mathbf{L}$, it is not difficult to show that $\mathbf{Z}(\omega^2)$ is symmetric [10]. Moreover, these properties can be used to show that Leung's theorem is also valid in this case, namely

$$\mathbf{M}(\omega^2) = -(\mathrm{d}/\mathrm{d}\omega^2)\mathbf{Z}(\omega^2),$$

where $\mathbf{M}(\omega^2) = \int_V \vec{\mathbf{P}}^t(\vec{\mathbf{x}}, \omega^2) \cdot \vec{\mathbf{P}}(\vec{\mathbf{x}}, \omega^2)\,\mathrm{d}v$ is the dynamic mass matrix of the structure.

A realistic example of the assumed displacement field imposed on the boundary (Γ_2) can be found in reference [10]. In this work, the assumed displacement field $\vec{\mathbf{u}}_\Gamma$ is a rigid-body displacement $\vec{\mathbf{u}}_\Gamma = \vec{\mathbf{r}}(t) + \vec{\boldsymbol{\omega}}(t) \wedge \vec{\mathbf{x}}$.

The displacement field $\vec{\mathbf{u}}_\Gamma$ is expressed in terms of the six parameters. $\mathbf{q} = (r_1, r_2, r_3, \omega_1, \omega_2, \omega_3)^t$, collecting the components of $\vec{\mathbf{r}}$ (rigid translation of Γ_2) and $\vec{\boldsymbol{\omega}}$ (rigid rotation of Γ_2) in the reference frame. The corresponding generalized force $\mathbf{Q}$ conjugated to $\mathbf{q}$ is defined by $\mathbf{Q} = (R_1, R_2, R_3, M_1, M_2, M_3)^t$ where (R_1, R_2, R_3) and (M_1, M_2, M_3) are the components of the resultant force and resultant torque exerted on the rigid boundary Γ_2

$$\vec{\mathbf{R}} = \int_{\Gamma_2} \sigma \vec{\mathbf{n}} \, \mathrm{d}S, \qquad \vec{\mathbf{M}} = \int_{\Gamma_2} \vec{\mathbf{x}} \wedge (\sigma \vec{\mathbf{n}}) \, \mathrm{d}S.$$

In this case, $\mathbf{Z}(\omega^2)$ is a symmetric (6×6) matrix giving the correspondence between the rigid displacement of the boundary Γ_2 and the resultant force and torque exerted on this boundary.

It is interesting to compare this result with the concept of impedance matrix in the case of Euler–Bernoulli beams. For Euler–Bernoulli beams, every cross-section behaves like a rigid body. So the boundaries in this case (i.e., the initial and final cross-sections of the beam) are rigid boundaries and the above theory applies. The impedance matrix of the Euler–Bernoulli beam gives the linear correspondence between the rigid displacements of the initial and final cross-sections of the beam in terms of the resultant force and resultant torque applied on those sections.

Another interesting result occurring from the above concept of impedance matrix for a 3-D continuum is that it is possible to obtain an expansion of this matrix in terms of an infinite set of vibrations modes of the structure [10, 12].

4. IMPEDANCE MATRIX FOR RECTANGULAR PLATES

Several attempts [13, 14] have been made to extend the concept of impedance matrix to the analytical vibrations analysis of plates. Only rectangular and circular plates have been previously considered. In contrast to the case of Euler–Bernoulli beams, even for rectangular or circular plates, no analytical solutions for the modal analysis of plates are available, except in the case of a very few particular boundary conditions [8, 18]. It is then hopeless to obtain an analytical form for the impedance matrix in this case.

4.1. GENERAL EQUATIONS

By considering a rectangular plate of dimension $2a \times 2b$, the transversal displacement $W(x, y, t)$ in classical Kirchhoff–Love theory can be defined by

$$\begin{aligned} &D(\Delta^2 W) + \mu \ddot{W} = 0, \qquad -a \leqslant x \leqslant a, \\ &D = Eh^3/12(1 - v^2), \quad -b \leqslant y \leqslant b, \end{aligned} \tag{13}$$

where E is the Young's modulus, v the Poisson ratio, h the plate thickness, μ the mass density

$$\Delta^2 W = \frac{\partial^4 W}{\partial x^4} + 2 \frac{\partial^4 W}{\partial x^2 \partial y^2} + \frac{\partial^4 W}{\partial y^4}, \qquad \ddot{W} = \frac{\partial^2 W}{\partial t^2}. \tag{14}$$

In Kirchhoff–Love theory the displacements on the boundaries $x = \pm a$ are defined by the column matrix

$$\tilde{\mathbf{d}}_1(y) = \begin{bmatrix} W(\pm a, y) \\ -(\partial W/\partial x)(\pm a, y) \end{bmatrix} \tag{15}$$

associated with a translation $W(\pm a, y)$ and a rotation $-(\partial W/\partial x)(\pm a, y)$ of the fiber $(x = \pm a, y, (-h/2) \leqslant z \leqslant h/2)$.

It means that each fiber behaves like a rigid body. Similarly, on the boundaries $y = \pm b$, the displacements are defined by the column matrix

$$\tilde{\mathbf{d}}_2(\mathbf{x}) = \begin{bmatrix} W(x, \pm b) \\ -(\partial W/\partial y)(x, \pm b) \end{bmatrix}. \tag{16}$$

The corresponding forces applied on these boundaries are also defined by a two-component column matrix. For the boundaries $x = \pm a$, the surface forces are defined by

$$\tilde{\mathbf{F}}_1(y) = \begin{bmatrix} V_1(\pm a, y) \\ M_1(\pm a, y) \end{bmatrix}, \tag{17}$$

where

$$V_1(\pm a, y) = -D[\partial^3 W/\partial x^3 + v^*(\partial^3 W/\partial x\, \partial y^2)]_{x=\pm a},$$

$$M_1(\pm a, y) = -D[\partial^2 W/\partial x^2 + v(\partial^2 W/\partial y^2)]_{x=\pm a}$$

$$v^* = 2 - v, \tag{18}$$

denote the effective shear force and the bending moment applied on the boundary. Similar results are obtained [18], for the two-component column matrix

$$\tilde{\mathbf{F}}_2(\mathbf{x}) = \begin{bmatrix} V_2(x, \pm b) \\ M_2(x, \pm b) \end{bmatrix} \tag{19}$$

associated with the surface forces applied on the boundaries $y = \pm b$.

As in the case of Euler–Bernoulli beam, the model corresponding to the assumption of Kirchhoff–Love theory of plates is connected to a rigid behavior of some parts of the boundaries. But in contrast to the case of Euler–Bernoulli beams, the displacements (and the corresponding forces) are not defined in terms of a finite set of parameters depending only on time. In plate theory, the displacement (and the forces) depend on the space variable describing the boundaries.

In order to avoid this dependence, the solution of the vibrations problem is obtained in terms of a finite set of base functions. A general solution of the equation (13), can be split up into four different symmetry cases

$$W(x, y) = W_{SS}(x, y) + W_{SA}(x, y) + W_{AS}(x, y) + W_{AA}(x, y). \tag{20}$$

The four symmetry cases are decoupled and therefore treated separately. One assumes that each one of these four displacements, denoted by $W(x, y)$ for reason of simplification, can be expressed in the form

$$W(x, y) = \sum_{n=1}^{N} X_n(x) f_n(y) + \sum_{n=1}^{N} Y_n(y) h_n(x), \tag{21}$$

where $f_n(y)$ and $h_n(x)$ are sinusoidal functions, their forms depend on the symmetries which are considered. $X_n(x)$ and $Y_n(y)$ are obtained from the conditions that $X_n(x) f_n(y)$ and $Y_n(y) h_n(x)$ are solutions of equation (13) in the frequency domain. It is not difficult to show according to the order of equation (13) and the conditions of symmetry, that each function $X_n(x)$ and $Y_n(y)$ depends on two constants. The solution $W(x, y)$ given by equation (21) depends on a set of $4N$ constants $\mathbf{c}$.

It is important to underline that the functions $f_n(y)$ and $h_n(y)$ are orthogonal functions, i.e.,

$$\int_0^b f_n(y) f_m(y) \, \mathrm{d}y = \mu_n \delta_{mn}, \qquad \int_0^a h_n(x) \, h_m(x) \, \mathrm{d}x = \lambda_n \delta_{mn}, \tag{22}$$

where δ_{mn} is the Kronecker delta function

4.1.1. *Projection method*

A first method to avoid the spatial dependence of the boundary conditions, is obtained by projection of the displacements and of the forces on these boundaries onto a set of functions $f_n(x)$ and $g_n(y)$. A vector of generalized displacements is defined by

$$\mathbf{q} = \begin{bmatrix} \displaystyle\int_0^b \frac{2}{\sqrt{\mu_m}} W(a, y) f_m(y) \, \mathrm{d}y \\ \displaystyle\int_0^b - \frac{2}{\sqrt{\mu_m}} \frac{\partial W(a, y)}{\partial x} f_m(y) \, \mathrm{d}y \\ \displaystyle\int_0^a \frac{2}{\sqrt{\lambda_m}} W(x, b) \; h_m(x) \, \mathrm{d}x \\ \displaystyle\int_0^a - \frac{2}{\sqrt{\lambda_m}} \frac{\partial W(x, b)}{\partial y} \; h_m(y) \, \mathrm{d}x \end{bmatrix}_{m=1,\ldots,N}. \tag{23}$$

Similarly, a vector which contains the generalized forces can be defined as

$$\mathbf{Q} = \begin{bmatrix} \displaystyle\int_0^b \frac{2}{\sqrt{\mu_m}} V_1(a, y) f_m(y) \, \mathrm{d}y \\ \displaystyle\int_0^b - \frac{2}{\sqrt{\mu_m}} M_1(a, y) f_m(y) \, \mathrm{d}y \\ \displaystyle\int_0^a \frac{2}{\sqrt{\lambda_m}} V_2(x, b) \; h_m(x) \, \mathrm{d}x \\ \displaystyle\int_0^a - \frac{2}{\sqrt{\lambda_m}} M_2(x, b) \; h_m(y) \, \mathrm{d}x \end{bmatrix}_{m=1,\ldots,N}. \tag{24}$$

Then, the generalized displacements $\mathbf{q}$ and the generalized forces $\mathbf{Q}$ on the boundaries are given by

$$\mathbf{q} = \tilde{\mathbf{D}}\mathbf{c}, \qquad \mathbf{Q} = \tilde{\mathbf{F}}\mathbf{c}. \tag{25, 26}$$

By elimination of $\mathbf{c}$, in equations (25) and (26), one obtains the impedance matrix which relates the generalized forces to the generalized displacements :

$$\mathbf{Q} = \mathbf{Z}\mathbf{q} = \tilde{\mathbf{F}}\tilde{\mathbf{D}}^{-1}\mathbf{q}. \tag{27}$$

It leads to the following definition of the impedance matrix $\mathbf{Z}$:

$$\mathbf{Z} = \tilde{\mathbf{F}}\tilde{\mathbf{D}}^{-1}. \tag{28}$$

4.1.2. *Energy method*

By rewriting solution (21) in the form

$$W(x, y) = g_1 W_1(x, y) + g_2 W_2(x, y) + \cdots + g_{4N} W_{4N}(x, y). \tag{29}$$

where $\mathbf{c} = (g_1 \cdots g_{4N})^{\mathrm{T}}$, the generalized displacements $\mathbf{d} = (\tilde{\mathbf{d}}_1 \tilde{\mathbf{d}}_2)^{\mathrm{T}}$ of the boundaries can be expressed in terms of the $4N$ parameters g_i by

$$\mathbf{d} = g_1 \mathbf{d}_1 + g_2 \mathbf{d}_2 + \cdots + g_{4N} \mathbf{d}_{4N}. \tag{30}$$

Similarly, the generalized forces $\mathbf{F} = (\tilde{\mathbf{F}}_1 \tilde{\mathbf{F}}_2)^t$ exerted on the boundaries can be written as

$$\mathbf{F} = g_1 \mathbf{F}_1 + g_2 \mathbf{F}_2 + \cdots + g_{4N} \mathbf{F}_{4N}. \tag{31}$$

The work done by the forces exerted on the boundary (Γ) of the plate is given by

$$T = \int_\Gamma \mathbf{d}^t \mathbf{F} \, \mathrm{d}s. \tag{32}$$

By introducing the notation

$$G_i = \int_\Gamma \mathbf{d}_i^t \mathbf{F} \, \mathrm{d}s \quad (i = 1, \ldots, 4N) \tag{33}$$

it follows that

$$G_i = \sum_{j=1}^{4N} Z_{e_{ij}} g_j, \tag{34}$$

where

$$Z_{e_{ij}} = \int_\Gamma \mathbf{d}_i^t \mathbf{F}_j \, \mathrm{d}s, \tag{35}$$

$$\mathbf{G} = \mathbf{Z}_e(\omega)\,\mathbf{c}, \quad \mathbf{G} = (G_1 \cdots G_n)^t, \quad \mathbf{c} = (g_1 \cdots g_n)^t \tag{36}$$

and $Z_e = (Z_{e_{ij}})$ is the impedance matrix which relates the parameters $\mathbf{c}$ and the corresponding parameters $\mathbf{G}$ conjugated to $\mathbf{c}$. It is not difficult to show that this new impedance matrix is symmetric.

In the following, a relation between the two impedance matrices $\mathbf{Z}$ and $\mathbf{Z}_e$ will be shown.

4.1.3. *Comparison of the two methods*

From equation (23) and properties (22) of the function $f_m(y)$, one deduces

$$W(a,y) = \sum_{m=1}^{N} q_{m1} \frac{1}{2\sqrt{\mu_m}} f_m(y), \qquad -\frac{\partial W}{\partial x}(a,y) = \sum_{m=1}^{N} q_{m2} \frac{1}{2\sqrt{\mu_m}} f_m(y),$$

$$W(x,b) = \sum_{m=1}^{N} q_{m3} \frac{1}{2\sqrt{\lambda_m}} h_m(x), \qquad \frac{\partial W}{\partial y}(x,b) = \sum_{m=1}^{N} q_{m4} \frac{1}{2\sqrt{\lambda_m}} h_m(x). \tag{37}$$

Similarly, from equations (24) and (22), one deduces:

$$V_x(a,y) = \sum_{m=1}^{N} Q_{m1} \frac{1}{2\sqrt{\mu_m}} f_m(y), \qquad M_x(a,y) = \sum_{m=1}^{N} Q_{m2} \frac{1}{2\sqrt{\mu_m}} f_m(y),$$

$$V_y(x,b) = \sum_{m=1}^{N} Q_{m3} \frac{1}{2\sqrt{\lambda_m}} h_m(x), \qquad M_y(x,b) = \sum_{m=1}^{N} Q_{m4} \frac{1}{2\sqrt{\lambda_m}} h_m(x). \tag{38}$$

It follows that

$$\begin{aligned}
4\int_0^b \tilde{\mathbf{F}}_1^t \tilde{\mathbf{d}}_1 \,\mathrm{d}y &= 4\int_0^b (W(a,y)\cdot V_x(a,y) + M_x(a,y)\cdot \Phi_x(a,y))\,\mathrm{d}y \\
&= 4\int_0^b \left(\sum_{m=1}^{N} q_{m1} \frac{1}{2\sqrt{\mu_m}} f_m(y) \right)\left(\sum_{p=1}^{N} Q_{p1} \frac{1}{2\sqrt{\mu_p}} f_p(y) \right) \mathrm{d}y \\
&\quad + 4\int_0^b \left(\sum_{m=1}^{N} q_{m2} \frac{1}{2\sqrt{\mu_m}} f_m(y) \right)\left(\sum_{p=1}^{N} Q_{p2} \frac{1}{2\sqrt{\mu_p}} f_p(y) \right) \mathrm{d}y \\
&= \sum_{m=1}^{N} (q_{m1} Q_{m1} + q_{m2} Q_{m2}).
\end{aligned} \tag{39}$$

One can easily show that similarly

$$4\int_0^a \tilde{\mathbf{F}}_2^t \tilde{\mathbf{d}}_2 \,\mathrm{d}x = \sum_{m=1}^{N} (q_{m3} Q_{m3} + q_{m4} Q_{m4}). \tag{40}$$

As a result, the work done by the generalized forces on the boundary (Γ) is given by

$$T = \mathbf{q}^{\mathrm{T}}\mathbf{Q} = \mathbf{c}^{\mathrm{T}}\tilde{\mathbf{D}}^{\mathrm{T}}\tilde{\mathbf{F}}\mathbf{c}, \tag{41}$$

where $\mathbf{q}$ and $\mathbf{Q}$ are the generalized displacements and the generalized forces defined by equations (23) and (24).

On the other hand, from equations (32) and (36)

$$T = \mathbf{c}^{\mathrm{T}}\mathbf{G} = \mathbf{c}^{\mathrm{T}}\mathbf{Z}_e\mathbf{c}, \tag{42}$$

it follows that

$$\mathbf{Z}_e = \tilde{\mathbf{D}}^{\mathrm{T}}\tilde{\mathbf{F}}. \tag{43}$$

From equations (28) and (43), one obtains

$$\mathbf{Z}_e = \tilde{\mathbf{D}}^{\mathrm{T}}\mathbf{Z}\tilde{\mathbf{D}}. \tag{44}$$

As the impedance matrix $\mathbf{Z}_e$ is symmetric, it is obvious that

$$\mathbf{Z}_e^{\mathrm{T}} = \mathbf{Z}_e = (\tilde{\mathbf{D}}^{\mathrm{T}}\mathbf{Z}\tilde{\mathbf{D}})^{\mathrm{T}} \tag{45}$$

and results in

$$\tilde{\mathbf{D}}^{\mathrm{T}}\mathbf{Z}^{\mathrm{T}}\tilde{\mathbf{D}} = \tilde{\mathbf{D}}^{\mathrm{T}}\mathbf{Z}\tilde{\mathbf{D}}, \tag{46}$$

so the impedance matrix $\mathbf{Z}$ is also symmetric. In contrast to the projection method, the impedance matrix $\mathbf{Z}_e$ is obtained without the inversion of any matrix.

In a first example, the method is applied to the vibrations of a rectangular plate simply supported on two opposite edges. In this particular case, the analytical solution of the problem is known [8, 18]. Then the more general case of rectangular plate with any boundary conditions is investigated.

4.2. RECTANGULAR PLATE WITH TWO OPPOSITE EDGES SIMPLY SUPPORTED (Figure 2)

The boundary conditions are

$$W(0,y) = W(a,y) = 0, \qquad M_x(0,y) = M_x(a,y) = 0. \tag{47}$$

By assuming that the solution is a solution of the Lévy type, the displacement solution has the form

$$W(x,y) = \sum_{m=1}^{N} Y_m(y)\sin\left(\frac{m\pi x}{a}\right). \tag{48}$$

which means that the boundary conditions are fulfilled whatever the value of y. So governing equation (13) becomes

$$Y_m''''(y) - 2\left(\frac{m\pi}{a}\right)^2 Y_m''(y) + \left\{\left(\frac{m\pi}{a}\right)^4 + \frac{\rho h\omega^2}{D}\right\} Y_m(y) = 0. \tag{49}$$

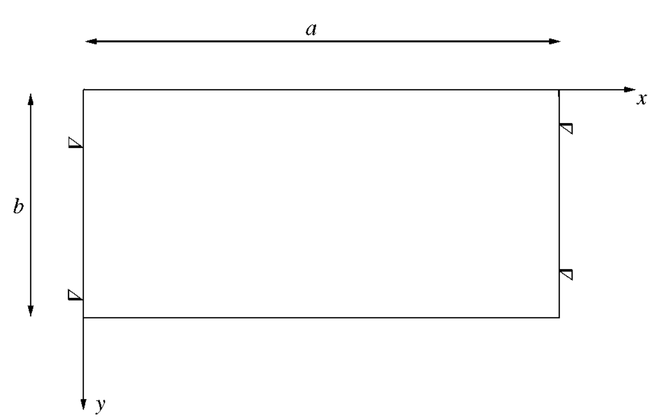


Figure 2. Rectangular plate with two opposite edges simply supported.

The solution of this equation is $Y_m(y) = \exp(\sigma_m y)$, with

$$\sigma_m^2 = (m\pi/a)^2 \mp \sqrt{\rho h\omega^2/D}. \tag{50}$$

Therefore, one obtains four values of σ_m, $\mp\sigma_{m1}$ and $\mp\sigma_{m2}$, where σ_{m1} and σ_{m2} are defined by the relations

$$\sigma_{m1} = \sqrt{(m\pi/a)^2 + \sqrt{\rho h\omega/D}}, \qquad \sigma_{m2} = \sqrt{(m\pi/a)^2 - \sqrt{\rho h\omega^2/D}}.$$

Two possibilities may be taken into consideration, that is, σ_{m1} and σ_{m2} are both real, or σ_{m1} is real and σ_{m2} is complex, so the solution has the form if $(m\pi/a)^2 > \sqrt{\rho h\omega^2/D}$.

$$Y_m(y) = A_m\cos(\sigma_{m1}y) + B_m\sin(\sigma_{m1}y) + C_m\cos(\sigma_{m2}y) + D_m\sin(\sigma_{m2}y)$$

and if

$$(m\pi/a)^2 < \sqrt{\rho h\omega^2/D},$$

$$Y_m(y) = A_m\cos(\sigma_{m1}y) + B_m\sin(\sigma_{m1}y) + C_m\cosh(\sigma_{m2}y) + D_m\sinh(\sigma_{m2}y),$$

where A_m, B_m, C_m, D_m are integration constants which depend on the boundary conditions. In the following, the cases σ_{m1} real and σ_{m2} complex are solved.

4.2.1. *Projection method*

The generalized displacements are given by the relationships

$$\mathbf{q}_m = \left\{\begin{array}{c} \displaystyle\int_0^a \frac{2}{\sqrt{a}} W(x,0)\sin\left(\frac{m\pi x}{a}\right)\mathrm{d}x \\ \displaystyle\int_0^a \frac{2}{\sqrt{a}}\left(-\frac{\partial W(x,0)}{\partial y}\right)\sin\left(\frac{m\pi x}{a}\right)\mathrm{d}x \\ \displaystyle\int_0^a \frac{2}{\sqrt{a}} W(x,b)\sin\left(\frac{m\pi x}{a}\right)\mathrm{d}x \\ \displaystyle\int_0^a \frac{2}{\sqrt{a}}\left(\frac{\partial W(x,b)}{\partial y}\right)\sin\left(\frac{m\pi x}{a}\right)\mathrm{d}x \end{array}\right\}. \tag{51}$$

In the same way, the generalized forces are defined by

$$\mathbf{Q}_m = \left\{\begin{array}{c} \displaystyle\int_0^a \frac{2}{\sqrt{a}} V_y(x,0)\sin\left(\frac{m\pi x}{a}\right)\mathrm{d}x \\ \displaystyle\int_0^a \frac{2}{\sqrt{a}} M_y(x,0)\sin\left(\frac{m\pi x}{a}\right)\mathrm{d}x \\ \displaystyle\int_0^a \frac{2}{\sqrt{a}} V_y(x,b)\sin\left(\frac{m\pi x}{a}\right)\mathrm{d}x \\ \displaystyle\int_0^a \frac{2}{\sqrt{a}} M_y(x,b)\sin\left(\frac{m\pi x}{a}\right)\mathrm{d}x \end{array}\right\}. \tag{52}$$

In this case, it is important to note that there is a total decoupling between terms for different values of m. This is due to the fact that the functions $\sin(m\pi x/a)$ are orthogonal to each function depending on x, occurring in equation (48). By eliminating the integration constants, one obtains a relationship between the generalized forces and the generalized displacements. By introducing an impedance matrix for one value of m,

$$\begin{Bmatrix} Q_{m1} \\ Q_{m2} \\ Q_{m3} \\ Q_{m4} \end{Bmatrix} = \mathbf{Z}_m \begin{Bmatrix} q_{m1} \\ q_{m2} \\ q_{m3} \\ q_{m4} \end{Bmatrix} \quad \text{with} \quad \mathbf{Z}_m = \tilde{\mathbf{F}}_m \tilde{\mathbf{D}}_m^{-1}. \tag{53}$$

Finally, one obtains the expression of the impedance matrix:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathbf{Z}_N \end{bmatrix}, \tag{54}$$

where

$$\mathbf{Z}_m = \frac{D}{b^3} \begin{bmatrix} F_{m_6} & -F_{m_4} b & F_{m_5} & Fm_3 b \\ -F_{m_4} b & F_{m_2} b^2 & -F_{m_3} b & F_{m_1} b^2 \\ F_{m_5} & -F_{m_3} b & F_{m_6} & F_{m_4} b \\ F_{m_3} b & F_{m_1} b^2 & F_{m_4} b & F_{m_2} b^2 \end{bmatrix}, \quad m = 1, 2, \ldots, N.$$

The functions F_{m_i} are defined by

$$F_{m_1} = -((\sigma_{m2}\sinh(\sigma_{m1}) - \sigma_{m1}\sinh(\sigma_{m2}))(\sigma_{m1}^2 + \sigma_{m2}^2))/\delta,$$

$$F_{m_2} = -((\sigma_{m1}\cosh(\sigma_{m1})\sin(\sigma_{m2}) - \sigma_{m2}\sinh(\sigma_{m1})\cos(\sigma_{m2}))(\sigma_{m1}^2 + \sigma_{m2}^2))/\delta,$$

$$F_{m_3} = -(\sigma_{m1}\sigma_{m2}(\sigma_{m1}^2 + \sigma_{m2}^2)(\cosh(\sigma_{m1}) - \cos(\sigma_{m2})))/\delta,$$

$$F_{m_4} = (\sigma_{m1}\sigma_{m2}[(\sigma_{m1}^2 + \sigma_{m2}^2)(\cosh(\sigma_{m1})\cos(\sigma_{m2}) - 1) + 2\sigma_{m1}\sigma_{m2}\sinh(\sigma_{m1})\sigma_{m2}])/\delta,$$

$$F_{m_5} = (\sigma_{m1}\sigma_{m2}(\sigma_{m1}^2 + \sigma_{m2}^2)(\sigma_{m2}\sin(\sigma_{m2}) + \sigma_{m1}\sinh(\sigma_{m1})))/\delta,$$

$$F_{m_6} = -(\sigma_{m1}\sigma_{m2}(\sigma_{m1}^2 + \sigma_{m2}^2)(\sigma_{m2}\cosh(\sigma_{m1})\sin(\sigma_{m2}) + \sigma_{m1}\sinh(\sigma_{m1})\sin(\sigma_{m2})))/\delta,$$

where

$$\delta = 2\sigma_{m1}\sigma_{m2}(\cosh(\sigma_{m1})\cos(\sigma_{m2}) - 1) - (\sigma_{m1}^2 + \sigma_{m2}^2)\sinh(\sigma_{m1})\sin(\sigma_{m2}).$$

It is obvious that the derived impedance matrix $\mathbf{Z}$ is symmetric.

4.2.2. *Energy method*

Formula (48) gives

$$W(x, y) = g_1 W_1(x, y) + g_2 W_2(x, y) + \cdots + g_{4N} W_{4N}(x, y), \tag{55}$$

where the parameters g_i are defined by

$$g_{1+4(m-1)} = A_m, \qquad g_{2+4(m-1)} = B_m, \qquad g_{3+4(m-1)} = C_m, \qquad g_{4+4(m-1)} = D_m,$$

$$m = 1, 2, \dots, N.$$

The displacement $\mathbf{d} = [W(x,0), -(\partial W/\partial y)(x,0), W(x,b), (\partial W/\partial y)(x,b)]^{\mathrm{T}}$, on the boundaries is given in terms of the $4N$ parameters g_i by

$$\mathbf{d} = g_1\mathbf{d}_1 + g_2\mathbf{d}_2 + \cdots + g_{4N}\mathbf{d}_{4N}. \tag{56}$$

In the same way, $\mathbf{F} = [V_y(x,0), M_y(x,0), -V_y(x,b), M_y(x,b)]$, is written as

$$\mathbf{F} = g_1\mathbf{F}_1 + g_2\mathbf{F}_2 + \cdots + g_{4N}\mathbf{F}_{4N}. \tag{57}$$

Let us introduce $\mathbf{G}$ conjugated to $\mathbf{g}$ computed by the work T and the impedance matrix which relates $\mathbf{G}$ to $\mathbf{g}$

$$G_i = \int_0^a \mathbf{d}_i^t\mathbf{F}\,\mathrm{d}x = \sum_j Z_{e_{ij}} g_j, \qquad Z_{e_{ij}} = \int_0^a \mathbf{d}_i^t\mathbf{F}_j\,\mathrm{d}s, \qquad \mathbf{G} = \mathbf{Z}_e(\omega)\,\mathbf{g}. \tag{58}$$

The energy impedance matrix has also been shown to be given by

$$\mathbf{Z}_e = \tilde{\mathbf{D}}^t\tilde{\mathbf{F}}. \tag{59}$$

4.3 GENERAL CASE

In the following, the general case of a rectangular plate with any boundary conditions is considered. To solve this problem, the solution of equation (13) is split, as in equation (20), into four particular solutions with special assumptions of symmetry or antisymmetry. Then one needs to consider only the case of double symmetry (the three other cases can be solved in the same way).

One assumes that the solution has the form

$$W(x,y) = \sum_{m=1}^{N} Y_m(y)\cos\left(\frac{m\pi x}{a}\right) + \sum_{m=1}^{N} X_m(x)\cos\left(\frac{m\pi y}{b}\right),$$

where $X_m(x)$ and $Y_m(y)$ are functions which satisfy (13) and the symmetry conditions. So each of them depends on two constants, as can be seen in the formulas

$$X_m(x) = A_m\cos\left(x\sqrt{\lambda^2 + (m\pi/b)^2}\right) + B_m\cos\left(x\sqrt{(m\pi/b)^2 - \lambda^2}\right),$$

$$Y_m(y) = C_m\cos\left(y\sqrt{\lambda^2 + (m\pi/a)^2}\right) + D_m\cos\left(y\sqrt{(m\pi/a)^2 - \lambda^2}\right),$$

where

$$\lambda = \omega^2\rho/D$$

In this case one assumes that $(m\pi/a)^2 - \lambda^2 > 0$, otherwise the function $\cos(y\sqrt{(m\pi/a)^2) - \lambda^2})$ must be changed by $\cosh(y\sqrt{|\lambda^2 - (m\pi/a)^2|})$.

In the same way, one can assume that $(m\pi/b)^2 - \lambda^2 > 0$ which results in the relations

$$\mathbf{d} = \begin{bmatrix} \tilde{\mathbf{d}}_1(a, y) \\ \tilde{\mathbf{d}}_2(x, b) \end{bmatrix} = \tilde{\mathbf{D}}\mathbf{c}, \qquad \mathbf{F} = \begin{bmatrix} \tilde{\mathbf{F}}_1(a, y) \\ \tilde{\mathbf{F}}_2(x, b) \end{bmatrix} = \tilde{\mathbf{F}}\mathbf{c},$$

where $\mathbf{c} = (A_m B_m C_m D_m)^{\mathrm{T}}$ $m = 1, \ldots, N$.

4.3.1. *Projection method*

This method uses the projection of displacements and forces onto the functions $\cos(m\pi y/b)$ and $\cos(m\pi x/a)$. By introducing $\mathbf{q}$ and $\mathbf{Q}$ defined by

$$\mathbf{q} = \left\{ \begin{matrix} \sqrt{\dfrac{8}{b}} \displaystyle\int_0^b W(a, y) \cos\left(\frac{m\pi y}{b}\right) \mathrm{d}y \\ \sqrt{\dfrac{8}{a}} \displaystyle\int_0^a W(x, b) \cos\left(\frac{m\pi x}{a}\right) \mathrm{d}x \\ \sqrt{\dfrac{8}{b}} \displaystyle\int_0^b -\frac{\partial W(a, y)}{\partial x} \cos\left(\frac{m\pi y}{b}\right) \mathrm{d}y \\ \sqrt{\dfrac{8}{a}} \displaystyle\int_0^a -\frac{\partial W(x, b)}{\partial y} \cos\left(\frac{m\pi y}{a}\right) \mathrm{d}x \end{matrix} \right\}_{m=1,\ldots,N}, \tag{60}$$

$$\mathbf{Q} = \left\{ \begin{matrix} \sqrt{\dfrac{8}{b}} \displaystyle\int_0^b V_1(a, y) \cos\left(\frac{m\pi y}{b}\right) \mathrm{d}y \\ \sqrt{\dfrac{8}{a}} \displaystyle\int_0^a V_2(x, b) \cos\left(\frac{m\pi x}{a}\right) \mathrm{d}x \\ \sqrt{\dfrac{8}{b}} \displaystyle\int_0^b M_1(a, y) \cos\left(\frac{m\pi y}{b}\right) \mathrm{d}y \\ \sqrt{\dfrac{8}{a}} \displaystyle\int_0^a M_2(x, b) \cos\left(\frac{m\pi x}{a}\right) \mathrm{d}x \end{matrix} \right\}_{m=1,\ldots,N}, \tag{61}$$

one notices, that contrary to the plate with two opposite edges simply supported, there is a coupling between the terms for different values of m.

The relation between the generalized displacements q_m and the constants $\mathbf{c}$ introduces a $(4, 4N)$ matrix $\tilde{\mathbf{D}}_m$; similarly, one obtains a relation between the generalized forces and the constants $\mathbf{c}$, using a $(4, 4N)$ matrix, $\tilde{\mathbf{F}}_m$,

$$\mathbf{q}_m = \tilde{\mathbf{D}}_m \mathbf{c}, \qquad \mathbf{Q}_m = \tilde{\mathbf{F}}_m \mathbf{c}. \tag{62}$$

Then by defining $\mathbf{q}$ as the vector of all the generalized displacements and $\mathbf{Q}$ as the vector of all the generalized forces

$$\mathbf{q} = (\mathbf{q}_i)_{i=1,\ldots,N}, \qquad \mathbf{Q} = (\mathbf{Q}_i)_{i=1,\ldots,N}. \tag{63}$$

One obtains the relation between **q** and **c**, **Q** and **c**, using $(4N, 4N)$ square matrices $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{F}}$. These matrices are block matrices defined by

$$\tilde{\mathbf{D}} = \begin{Bmatrix} \tilde{\mathbf{D}}_1 \\ \vdots \\ \tilde{\mathbf{D}}_N \end{Bmatrix} \qquad \tilde{\mathbf{F}} = \begin{Bmatrix} \tilde{\mathbf{F}}_1 \\ \vdots \\ \tilde{\mathbf{F}}_N \end{Bmatrix}.$$

Then one deduces

$$q = \tilde{\mathbf{D}}\mathbf{c}, \qquad \mathrm{Q} = \tilde{\mathbf{F}}\mathbf{c}. \tag{64}$$

By eliminating **c**, these equations, the impedance matrix which relates generalized forces to generalized displacements is obtained as

$$\mathbf{Q} = \mathbf{Z}\mathbf{q} = \tilde{\mathbf{F}}\tilde{\mathbf{D}}^{-1}\mathbf{q}.$$

4.3.2. *Energy method*

The work done by the forces on the boundaries, is given by

$$T = 4\left(\int_0^b \tilde{\mathbf{F}}_1^t \tilde{\mathbf{d}}_1 \mathrm{d}y + \int_0^a \tilde{\mathbf{F}}_2^t \tilde{\mathbf{d}}_2 \, \mathrm{d}x\right). \tag{65}$$

The relationships

$$\tilde{\mathbf{d}}_1(a, y) = \mathbf{\Delta}_1(y)\,\mathbf{c}, \qquad \tilde{\mathbf{d}}_2(x, b) = \mathbf{\Delta}_2(x)\,\mathbf{c}, \tag{66}$$

$$\tilde{\mathbf{F}}_1(a, y) = \mathbf{\Phi}_1(y)\,\mathbf{c}, \qquad \tilde{\mathbf{F}}_2(x, b) = \mathbf{\Phi}_2(x)\,\mathbf{c}, \tag{67}$$

are defined where $\mathbf{\Delta}_1(y)$, $\mathbf{\Delta}_2(x)$, $\mathbf{\Phi}_1(y)$, $\mathbf{\Phi}_2(x)$ are matrices of dimension $(2, 4N)$.

Inserting (66) and (67) into equation (32), one can write

$$T = \mathbf{c}^t\left(4\int_0^b \mathbf{\Delta}_1^t \mathbf{\Phi}_1 \, \mathrm{d}y + 4\int_0^a \mathbf{\Delta}_2^t \, \mathbf{\Phi}_2 \, \mathrm{d}x\right)\mathbf{c} = \mathbf{c}^t \, \mathbf{Z}_e \mathbf{c}.$$

The impedance matrix is given by the formula

$$\mathbf{Z}_e = 4\int_0^b \mathbf{\Delta}_1^t \mathbf{\Phi}_1 \, \mathrm{d}y + 4\int_0^a \mathbf{\Delta}_2^t \, \mathbf{\Phi}_2 \, \mathrm{d}x. \tag{68}$$

By introducing **C** conjugated to **c** computed by the work T done by the boundary forces

$$\mathbf{C} = \mathbf{Z}_e \, \mathbf{c}.$$

5. NUMERICAL EXAMPLES

In the above study, the general solution of the vibrations equation of a rectangular plate, is expressed as a function of a truncated Lévy series:

$$W(x, y) = g_1 W_1(x, y) + g_2 W_2(x, y) + \cdots + g_{4N} W_{4N}(x, y) \tag{69}$$

with

$$g_{1+4(m-1)} = A_m, \qquad g_{2+4(m-1)} = B_m, \qquad g_{3+4(m-1)} = C_m, \qquad g_{4+4(m-1)} = D_m,$$
$$m = 1, 2, \ldots, N.$$

This assumption allows one to express the generalized displacements and the generalized forces on the edges of the plate as a function of the parameters $(g_1, \ldots, g_{4N})$. One obtains

$$\mathbf{d} = \tilde{\mathbf{D}}(s, \omega)\,\mathbf{c}, \qquad \mathbf{F} = \tilde{\mathbf{F}}(s, \omega)\,\mathbf{c},$$

where

$$\mathbf{c} = (g_1 \cdots g_{4N})^{\mathrm{T}}. \tag{70}$$

The determination of the eigenvalues of the plate, corresponding to the different boundary conditions, is equivalent to the determination of the parameters $(g_1, \ldots, g_{4N})$ in order to fulfill these boundary conditions. For the vibrations of a plate with clamped boundaries, the constants $\mathbf{c}$ are defined by the condition $\mathbf{d} = \tilde{\mathbf{D}}(s, \omega)\,\mathbf{c} = 0$ on (Γ). For the vibrations of a plate with free boundaries, $\mathbf{c}$ are defined by $\mathbf{F} = \tilde{\mathbf{F}}(s, \omega)\,\mathbf{c} = 0$ on (Γ).

As it is not possible to obtain the analytical solution of this problem, the boundary conditions are projected onto a finite set of base functions. To obtain the vibration modes of the plate with clamped boundaries, one replaces the condition $\mathbf{d} = 0$ with $\int_\Gamma \tilde{\mathbf{F}}^t \mathbf{d}\, \mathrm{d}s = 0$. To obtain the vibration modes of the plate with free boundaries, one replaces the condition $\mathbf{F} = 0$ with $\int_\Gamma \tilde{\mathbf{D}}^t \mathbf{F}\, \mathrm{d}s = 0$ which is equivalent to

$$\mathbf{Z}_e(\omega)\,\mathbf{c} = 0, \qquad \mathbf{Z}_e = \int_\Gamma \tilde{\mathbf{D}}^t\,\tilde{\mathbf{F}}\,\mathrm{d}s. \tag{71}$$

In this case, the eigenpulsations are the roots of the equation

$$\det(\mathbf{Z}_e(\omega)) = 0. \tag{72}$$

In the following, one uses this method for the modal analysis of a rectangular plate with two edges simply supported, for a free–free plate and for a plate with two clamped edges.

5.1. RECTANGULAR PLATE WITH TWO OPPOSITE EDGES SIMPLY SUPPORTED

One considers a rectangular plate (Figure 3) simply supported on the edge $x = 0$ and on the edge $x = a$, with dimensions $a = 25{\cdot}4$ cm, $b = 38{\cdot}1$ cm, $h = 3{\cdot}175$ mm, $E = 68\,948$ MPa, $\rho = 2700\,\mathrm{kg/m^3}$, $\nu = 0{\cdot}333$, and searches the eigenfrequencies which are in the bandwidth (2300–3000 Hz) (Table 1). The results obtained from the energy method are compared with those given by the projection method by Fleuret [15], and with those of the finite element code CASTEM.

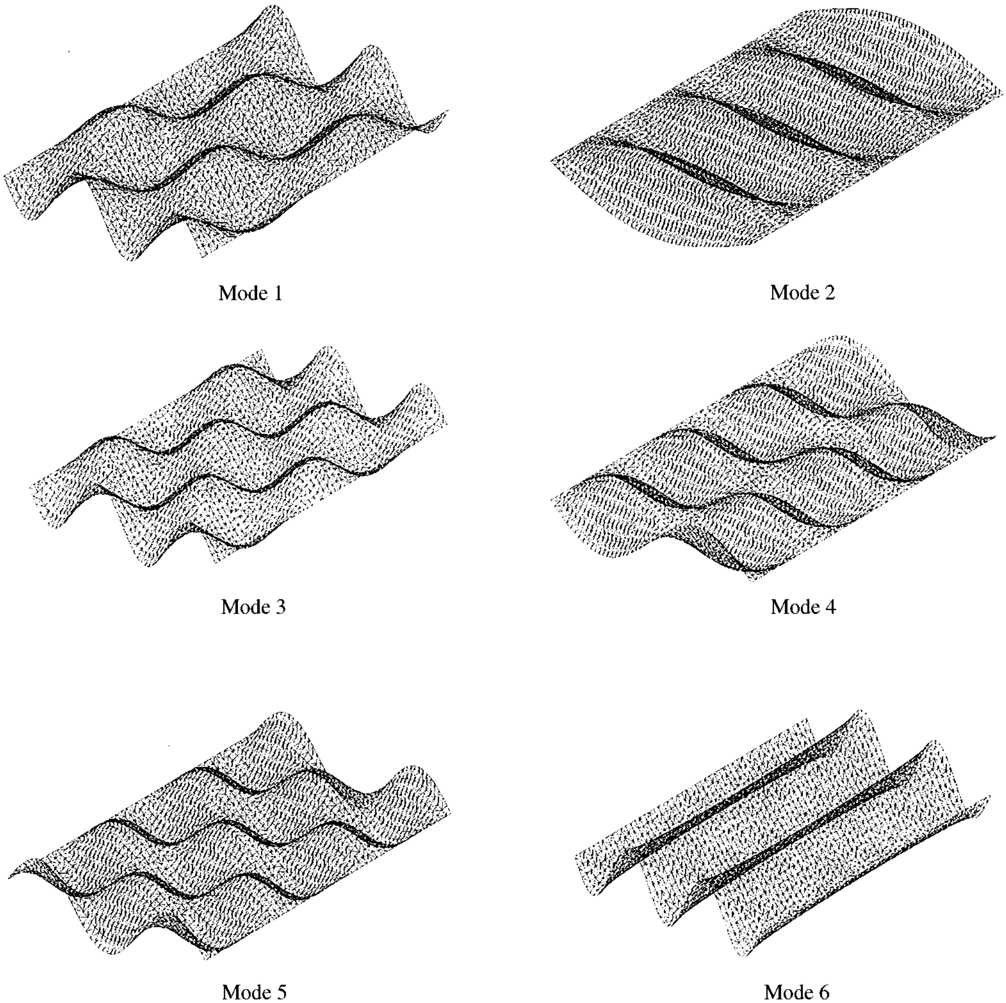


Figure 3. Eigenforms of a simply supported rectangular plate.

Table 1

Comparison of results (in Hz) for a rectangular plate with two opposite edges simply supported

Value of m	Energy	Projections	CASTEM
4	2337·39	2338·40	2338·862
1	2381·98	2382·98	2383·802
4	2683·69	2683·98	2682·774
2	2786·71	2785·84	2782,465
3	2793·36	2797·67	2792·572
5	2971·09	2971·45	2971·299

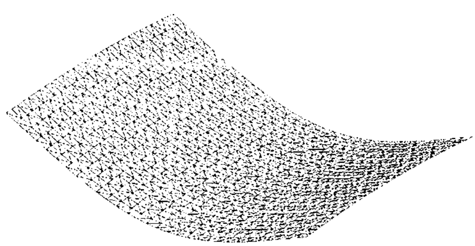
5.2. RECTANGULAR PLATE WITH FREE BOUNDARIES

The modes with the double symmetry property of a rectangular plate (Table 2) having as dimensions (Figure 4) $a = 45\,\mathrm{cm}$, $b = 30\,\mathrm{cm}$, $h = 0{\cdot}3\,\mathrm{cm}$, $E = 70\,000\,\mathrm{MPa}$, $\rho = 2790\,\mathrm{kg/m^3}$, $\nu = 0{\cdot}333$ are considered. The results obtained from the energy method are compared to those given by the projection and the finite element methods.

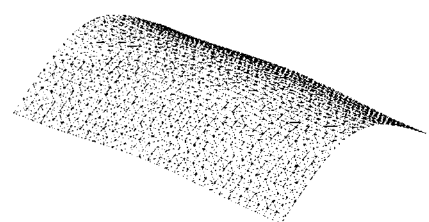
TABLE 2

Comparison of results (*in* Hz) for a rectangular plate with free boundaries

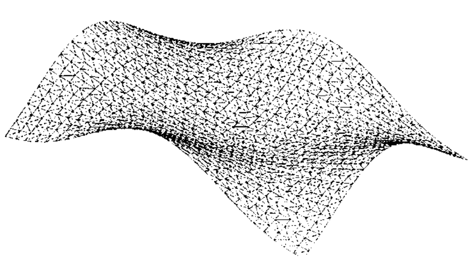
Energy	Projections	CASTEM
76·29	76·56	76·622
178·89	179·97	180·05
350·05	351·92	352·57
431·42	432·57	434·16



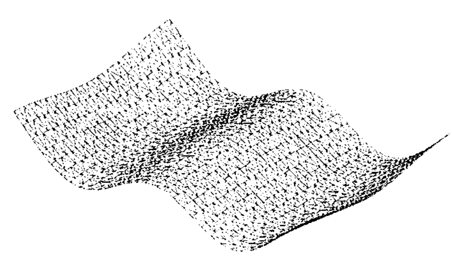
First symmetric–symmetric mode



Second symmetric–symmetric mode



Third symmetric–symmetric mode



Fourth symmetric–symmetric mode

Figure 4. Symmetric–symmetric eigenforms of a free rectangular plate.

5.3. RECTANGULAR PLATE WITH TWO OPPOSITE EDGES CLAMPED

The modes with the double symmetry property of a rectangular plate (Table 3) having the same dimensions as before is considered. The results are compared with the finite element code CASTEM (Figure 5). Good agreement for the three kinds of boundary conditions, has been obtained.

TABLE 3

Comparison of results (in Hz) for a rectangular plate with two opposite edges clamped

Energy	CASTEM
79·22	79·61
426·26	431·80
705·05	715·06
1061·47	1069·49

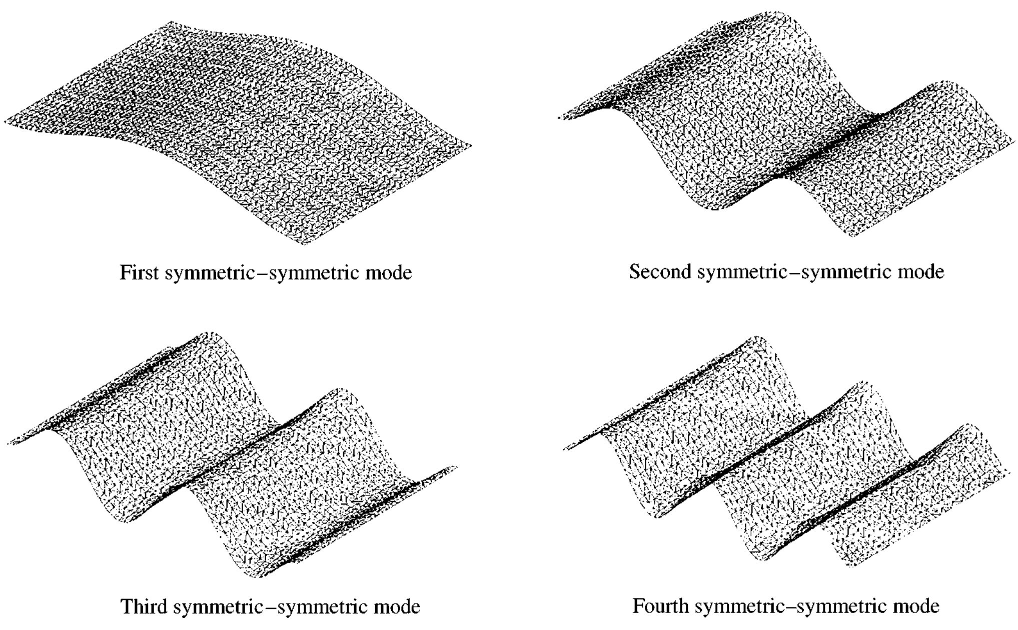


Figure 5. Symmetric–symmetric eigenforms of a rectangular plate with two opposite edges clamped.

6. CONCLUSION

In this paper, a new concept of an impedance matrix for the analysis of a continuous system is introduced, this formulation being consistent with the concept of impedance matrix of discretized structures and beam–column structures. The method is applied to rectangular plates leading to the solution of vibration problem for boundary solutions other than the simply supported ones.

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